

ADA064427

CONTROL ANALYSIS CORPORATION
800 Welch Road
Palo Alto, California

Technical Report No. 712-1

January 1979

12

DDC FILE COPY

ON TOTAL SOJOURN TIME
IN
ACYCLIC JACKSON NETWORKS*

by

Austin J. Lemoine

(Signature) 1473

Approved for public release;
Distribution Unlimited

DDC
RECEIVED
FEB 9 1979
(Signature) E

*This work was supported by the Office of Naval Research under contract No. N00014-78-C-0712 (NR-047-106).

Reproduction in whole or in part is permitted for any purpose of the United States Government.

79 5 120

ABSTRACT

In this paper, we compute the mean of the equilibrium (steady-state) total sojourn time distribution for the important class of infinite capacity acyclic Jackson networks with a single server at each node. In addition, for those acyclic Jackson networks with a 'tree-like' structure, we derive the Laplace transform of the equilibrium total sojourn time distribution and then give a simple recursive procedure for computing the higher moments of the distribution. Such basic results should prove helpful in testing procedures for simulation output analysis of infinite capacity open networks of queues.

| | |
|---------------------------|---|
| ACCESSION for | |
| NTIS | Write Station <input checked="" type="checkbox"/> |
| DDC | Bull Section <input type="checkbox"/> |
| UNANNOUNCED | <input type="checkbox"/> |
| JUSTIFICATION | |
| BY | |
| DISTRIBUTION/AVAILABILITY | |
| Dist. | Avail. and Z. |
| A | |

0. INTRODUCTION

Queueing network models abound in applications, but despite the immense practical importance of such models the body of available and useful results for networks of queues is far from satisfactory; see Lemoine [9], [10] for a comprehensive review of available equilibrium results and weak convergence results for networks of queues. For studying many queueing network models simulation would appear to be the only practical recourse at the present time. The regenerative method for simulation analysis has been applied to the study of passage time problems in closed networks of queues and finite capacity open networks by Iglehart and Shedler [4], [5], [6]. The report of Lavenberg [8] discusses application of the regenerative method to simulations of closed networks of queues. However, the regenerative method would appear to be inappropriate for the large and important body of infinite capacity open network models; such queueing networks are probably too complicated to return often enough to some "regenerative condition" from which the entire network starts afresh probabilistically. Nevertheless, any candidate procedure for simulation analysis of infinite capacity open networks of queues requires a basic model, and theoretical results for such a model to serve as a testing ground for the procedure. For example, the M/M/1 queue has been invaluable as a testing ground for the development of the various aspects of the regenerative method. For infinite capacity open networks an appropriate test model is the classical Markovian network system of Jackson [7] with a single server at each node. And, for infinite capacity open networks an important characteristic of system performance is the total

sojourn time (total response time) distribution for typical customers in the network. In this paper, therefore, we compute the mean of the equilibrium (steady-state) total sojourn time distribution for the important class of infinite capacity acyclic Jackson networks with a single server at each node. In addition, for those acyclic Jackson networks with a "tree-like" structure, we derive the Laplace transform of the equilibrium total sojourn time distribution and then give a simple recursive procedure for computing the higher moments of the distribution. Such basic results should prove helpful in testing procedures for simulation output analysis of infinite capacity open networks of queues.

1. THE BASIC MODEL AND STATEMENT OF THE RESULT

The model of interest here is a Markovian network of queues of the type introduced in the classical paper of Jackson [7]. There are N nodes with node i having a single-server, a first-come-first-served queue discipline, and a waiting room of unlimited capacity. The external input stream to node i is Poisson with rate λ_i , and these external input streams are assumed to be independent. The service times at node i are independent and have a common exponential distribution with parameter μ_i , and are independent of all customer arrivals at node i . A customer leaving node i is immediately and independently routed to node j with probability p_{ij} , and the customer departs the system from node i with probability $q_i = 1 - \sum_{j=1}^N p_{ij}$.

The state of the network at time t is taken to be

$$C(t) = (c_1(t), c_2(t), \dots, c_N(t)) \quad (1)$$

where $c_i(t)$ is the number of customers at node i at time t . Given the independent Poisson external input streams, the exponential service times at the various nodes, and the independent routing scheme, it follows that $\{C(t), t \geq 0\}$ is a Markov process with stationary transition probabilities. In this paper we are interested in the total sojourn times of customers in the network when the process $\{C(t), t \geq 0\}$ has an equilibrium (or limiting) distribution. In particular, we are interested in the distribution of total sojourn time under the equilibrium Markov

queue-lengths vector process

$$\{C(t), -\infty < t < +\infty\} . \quad (2)$$

This equilibrium process (when it exists) will henceforth often be denoted by $C(\cdot)$. And, this equilibrium process exists if (and only if) the "traffic intensity" is less than one at each node in the network. Traffic intensity in this network setting means the following. Let P be the $N \times N$ matrix of the p_{ij} 's and let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ be the row vector solution of the "traffic equation"

$$\alpha = \lambda + \alpha P \quad (3)$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$. Since customers eventually leave the system each entry of the matrix P^m converges to 0 as $m \rightarrow \infty$, so that the matrix $I - P$ is invertible and (3) has a unique solution given λ . In row form (3) is equivalent to

$$\alpha_i = \lambda_i + \sum_{j=1}^N p_{ij} \alpha_j , \quad i = 1, 2, \dots, N . \quad (3a)$$

This is a balance or conservation equation which says that the equilibrium rate of flow through node i , α_i , is the sum of the external input rate, λ_i , and the total rate of internal transfers to node i , $\sum_{j=1}^N p_{ji} \alpha_j$. Then, if the "traffic intensity"

$$\rho_i = \alpha_i / \mu_i < 1 \quad (4)$$

for each i , the queue-lengths vector process has a unique equilibrium distribution π , where for $C = (c_1, c_2, \dots, c_N)$ a N -tuple of non-negative integers

$$\pi(C) = \prod_{i=1}^N \psi_i(c_i) \quad (5)$$

with $\psi_i(c_i) = (1 - \rho_i) \rho_i^{c_i}$. Thus, when (and only when) the condition (4) holds for each node in the network, the random vector $C(t)$, vis`a vis the equilibrium process $C(\cdot)$, has distribution π for each t in $(-\infty, +\infty)$. Another way of saying this is the following. Let p denote the (stationary) transition probability function of the Markov queue-lengths vector process; that is, if $-\infty < s < t < +\infty$ and C and D are possible states then

$$P\{C(t) = D | C(s) = C\} = p(D, C, t - s) \quad (6)$$

on the set $\{C(s) = D\}$ with probability one. Then π is the equilibrium distribution if (and only if)

$$\sum_D \pi(D) p(D, C, y) = \pi(C) \quad (7)$$

for all y in $(0, \infty)$ and all states C .

Now, let the random epochs of external customer arrivals in the equilibrium process $C(\cdot)$ be

$$\dots < t_{-2} < t_{-1} < 0 < t_1 < t_2 \dots \quad (8)$$

The points $\{t_n, n = \pm 1, \pm 2, \dots\}$ are the superposition of N independent Poisson processes on $(-\infty, +\infty)$ with intensities $\lambda_1, \lambda_2, \dots, \lambda_N$; and the customer arriving at epoch t_n enters the network via node i independently with probability $\lambda_i/(\lambda_1 + \lambda_2 + \dots + \lambda_N)$ for all n and i . Let

$$t_n + T_n \quad (9)$$

denote the random epoch of departure from the network for the customer arriving at epoch t_n , so that T_n is the total sojourn time in the network for customer n . Since the network is in equilibrium the sojourn times $\{T_n, n = \pm 1, \pm 2, \dots\}$ have a common distribution which we denote by H . In this paper we show that if the network is acyclic under \mathcal{P} then the mean of the distribution H , say $E\{T\}$, is given by

$$E\{T\} = \left(\sum_{i=1}^N \lambda_i \right)^{-1} \sum_{i=1}^N \frac{\rho_i}{1 - \rho_i} \quad (10a)$$

Moreover, if the acyclic network has a "tree-like" structure then we show that the Laplace transform, say h , of the equilibrium total sojourn time distribution H is given by

$$h(\theta) = \bar{\lambda} [\mathcal{J}(\theta) - \mathcal{P}]^{-1} \mathbf{q} \quad (10b)$$

for $\theta \geq 0$, where $\mathcal{J}(\theta)$ is a diagonal matrix whose i th diagonal entry is $(\mu_i - \alpha_i + \theta)/(\mu_i - \alpha_i)$, $\bar{\lambda}$ is a row vector whose i th entry

is $\lambda_i / (\lambda_1 + \lambda_2 + \dots + \lambda_N)$, and q is a column vector whose i th entry is q_i . We also give a simple recursive procedure for computing all the moments of the distribution with transform given by (10b). The transform result (10b) extends a result of Reich [11] for single-server Markovian queues in tandem.

These results were derived by a heuristic argument in [9]. The argument given here proceeds in stages, each of which comprises a separate section of the paper. The discussion in Sections 2 and 3, and the first part of Section 4 does not require that the network be acyclic. From thereon, however, acyclic structure plays a crucial role.

2. THE PROCESS $\{C(t), -\infty < t < +\infty\}$

In this section we observe that the process $\{C(t), -\infty < t < +\infty\}$ is a Markov process in equilibrium, with equilibrium distribution π and (stationary) transition probability function p .

Let the process $\{C(t), -\infty < t < +\infty\}$ be defined on the probability space (Ω, \mathcal{F}, P) . Without loss of generality we take $C(\cdot)$ to have sample paths which are constant except for isolated jumps, are right continuous, and have left limits, all with probability one. With this setup the process $C(\cdot)$ has no fixed points of discontinuity with probability one, so that $P\{C(t) = C(t-)\} = 1$ for each t in $(-\infty, +\infty)$. Hence, in order to establish that the process $\{C(t), -\infty < t < +\infty\}$ has the stated properties, it suffices to show that for arbitrary $-\infty < s < t < +\infty$ we have

$$P\{C(t-) = C|C(s-), u \leq s\} = p(C(s-), t - s, C) \quad (11)$$

for all possible states C with probability one.

Now, given the structure of the sample paths of $C(\cdot)$ we clearly have

$$P\{C(t-) = C|C(s-), u \leq s\} = P\{C(t-) = C|C(s), u < s\}$$

with probability one. Let $\epsilon_m = 1/2^m$ for $m = 1, 2, \dots$. Since $C(\cdot)$ is a Markov process

$$P\{C(t - \epsilon_m) = C|C(s), u \leq s - \epsilon_m\} = p(C(s - \epsilon_m), C, t - s)$$

for all m , with probability one. By Theorem 9.4.8 in Chung [3]

$$\lim_{m \rightarrow \infty} P\{C(t - \epsilon_m) = C\{C(u), u \leq s - \epsilon_m\} = P\{C(t-) = C\{C(u), u < s\}$$

with probability one. Moreover, we clearly have $p(C(s - \epsilon_m), C, t - s) \rightarrow p(C(s-), C, t - s)$ as $m \rightarrow \infty$, with probability one. The proof of (11) is complete.

3. EXTERNAL ARRIVALS AND EXTERNAL DEPARTURES

As in (8) let $\{t_n, n = \pm 1, \pm 2, \dots\}$ be the random epochs of external arrivals to the network. Similarly, let

$$\dots < d_{-2} < d_{-1} < 0 < d_1 < d_2 < \dots \quad (12)$$

be the random epochs of external departures from the network. In this section we observe that the random vectors $\{C(t_n^-), n = \pm 1, \pm 2, \dots\}$ and $\{C(d_n), n = \pm 1, \pm 2, \dots\}$ are identically distributed and have π for their common distribution.

Let $\{q(C), q(C,D)\}$ denote the transition rates for the process $C(\cdot)$. That is, upon entering state C , for example, the process remains there for a random time having an exponential distribution with parameter $q(C)$, and upon leaving state C the process goes to some state $D \neq C$ with probability $q(C,D)/q(C)$. The equilibrium distribution π satisfies the "balance equation"

$$\pi(C) q(C) = \sum_{D \neq C} \pi(D) q(D,C) \quad (13)$$

for all states C . For $1 \leq i \leq N$ let E_i denote the N -vector with all components zero except for a 1 in component i . If $C(t) = C$, then the next transition will be either to state $C + E_i$ (external arrival at node i), to state $C - E_i$ (external departure from node i), or to state $C - E_i + E_j$ for $j \neq i$ (transfer from node i to node j). Thus for state C

$$q(C) = \sum_{i=1}^N q(C, C + E_i) + \sum_{i=1}^N q(C, C - E_i) \\ + \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N q(C, C - E_i + E_j) .$$

In addition to satisfying (13), the distribution π also satisfies the "partial balance equation"

$$\pi(C) \sum_{i=1}^N q(C, C + E_i) = \sum_{i=1}^N \pi(C + E_i) q(C + E_i, C) \quad (14)$$

for each state C ; cf. [9].

Consider the "reversed process" $\{C(-t), -\infty < t < +\infty\}$. The reversed process is also a Markov process in equilibrium with the same distribution π . The transition rates $\{q'(C), q'(C, D)\}$ for the reversed process are given by

$$\pi(C) q(C, D) = \pi(D) q'(D, C) \quad (15)$$

and

$$q(C) = q'(C) \quad (16)$$

for all states C and D .

If we now apply (15) to each term on the right side of (14), divide both sides of (14) by $q(C)$, and then invoke (16), we obtain

$$\pi(C) \sum_{i=1}^N \frac{q(C, C+E_i)}{q(C)} = \pi(C) \sum_{i=1}^N \frac{q'(C, C+E_i)}{q'(C)} \quad (17)$$

for each possible state C . In the reversed process the transitions from state C to state $C + E_i$, $1 \leq i \leq N$, are registered at the epochs $\{(-d_n) +, n = \pm 1, \pm 2, \dots\}$. Note that the sample paths of the reversed process are left continuous while those of $C(\cdot)$ are right continuous. Thus, since π is the common distribution of $C(\cdot)$ and of the reversed process, and since the state space of $\{C(t_n-), n = \pm 1, \pm 2, \dots\}$ and $\{C((-d_n)+), n = \pm 1, \pm 2, \dots\}$ coincide with the state space of $C(\cdot)$ and the reversed process, we conclude from (17) that the random vectors $\{C(t_n-), n = \pm 1, \pm 2, \dots\}$ and $\{C((-d_n)+), n = \pm 1, \pm 2, \dots\}$ are identically distributed and have π for their common distribution. Moreover, since $C(\cdot)$ and the reversed process are equilibrium processes defined over the time interval $(-\infty, +\infty)$, the vectors $\{C(d_n), n = \pm 1, 2, \dots\}$ and $\{C((-d_n)+), n = \pm 1, \pm 2, \dots\}$ are identically distributed.

4. EXIT SETS AND NODES

Let V be a non-empty set of nodes, that is, V is a non-empty subset of $\{1, 2, \dots, N\}$. Let V^c denote the complement of V in $\{0, 1, 2, \dots, N\}$ where node 0 denotes the network terminus or sink. We say that V is an exit set if $p_{kr} = 0$ for each node r in V and each node k in V^c , where $p_{r0} \equiv q_r$ for r in V . Equivalently, V is an exit set if upon leaving V there is no path in the network leading back to V . Note that $\{1, 2, \dots, N\}$ is an exit set.

In the equilibrium process $C(\cdot)$, let $E_{rk}(s, t]$ be the number of customers who depart node r and arrive instantaneously at node k over the time interval $(s, t]$. If V is an exit set then by results of Beutler and Melamed [1] and Walrand and Varaiya [14] the streams E_{jk} , j in V and k in V^c , are mutually independent Poisson processes with respective intensities $\alpha_j p_{jk}$. In particular, since $\{1, 2, \dots, N\}$ is an exit set, the external departure streams E_{i0} , $1 \leq i \leq N$, are mutually independent Poisson process on $(-\infty, +\infty)$ with respective intensities $\alpha_i q_i$.

From here on, suppose that the network is acyclic under \mathcal{P} . For node i , $1 \leq i \leq N$, let V_i be the set of all nodes r from which i is accessible under \mathcal{P} . Observe that node i is not a member of V_i and that

$$\alpha_i = \lambda_i + \sum_{r \in V_i} \alpha_r p_{ri}.$$

Let $V_{(i)}$ consist of node i together with the nodes of V_i . Then

both V_i and $V_{(i)}$ are exit sets for $i = 1, 2, \dots, N$.

Let W_i denote the complement of V_i in $\{1, 2, \dots, N\}$. Then the streams E_{rk} , r in V_i and k in W_i , are mutually independent Poisson processes with respective intensities $\alpha_r p_{rk}$. Thus, since $C(\cdot)$ is in equilibrium, the set of nodes W_i is a Jackson network in equilibrium with Poisson external input intensities

$$\hat{\lambda}_k = \lambda_k + \sum_{V_i} \alpha_r p_{rk}$$

for k in W_i . Also,

$$\hat{\alpha}_k = \hat{\lambda}_k + \sum_{W_i} \hat{\alpha}_r p_{rk} = \alpha_k$$

for k in W_i . In particular, note that $\hat{\lambda}_i = \alpha_i$, so that if

$$\dots < t_{-2i} < t_{-1i} < 0 < t_{1i} < t_{2i} < \dots \quad (18)$$

are the random epochs of pooled customer arrivals at node i , external arrivals plus internal transfers from other nodes, then the points $\{t_{ni}, n = \pm 1, \pm 2, \dots\}$ form a Poisson process on $(-\infty, +\infty)$ with intensity α_i . The equilibrium distribution for the Jackson network W_i , say $\hat{\pi}$, is given by

$$\hat{\pi}(\hat{C}) = \prod_{W_i} \psi_k(c_k)$$

where $\hat{C} = (c_k, k \text{ in } W_i)$ is a vector of non-negative integers.

Likewise, if W_i^* is the complement of $V_{(i)}$ in $\{1, 2, \dots, N\}$, then the streams E_{rj} , $r \text{ in } V_{(i)}$ and $j \text{ in } W_i^*$, are mutually independent Poisson processes with respective intensities $\alpha_r p_{rj}$. The set of nodes W_i^* is thus a Jackson network in equilibrium with Poisson external input intensities

$$\lambda_j^* = \lambda_i + \sum_{V_{(i)}} \alpha_r p_{rj}$$

for $j \text{ in } W_i^*$, and

$$\alpha_j^* = \lambda_j^* + \sum_{W_i^*} \alpha_r^* p_{rj} = \alpha_j$$

for $j \text{ in } W_i^*$. The equilibrium distribution for the Jackson network W_i^* , say π^* , is given by

$$\pi^*(C^*) = \prod_{W_i^*} \psi_j(c_j)$$

where $C^* = (c_j, j \text{ in } W_i^*)$ is a vector of non-negative integers.

For $-\infty < t < +\infty$ let

$$\hat{C}(t) = (c_k(t), k \text{ in } W_i)$$

and

$$C^*(t) = (c_j(t), j \text{ in } W_i^*) .$$

Note that $\hat{C}(t) = (c_i(t), C^*(t))$ and that if $\hat{C} = (c, C^*)$ then $\hat{\pi}(\hat{C}) = \psi_i(c)\pi^*(C^*)$. Thus, if $\{\hat{t}_{ni}, n = \pm 1, \pm 2, \dots\}$ are the points in the Poisson process of external arrivals to the Jackson network W_i , then by Section 3 we have

$$P\{\hat{C}(\hat{t}_{ni}-) = (c, C^*)\} = \psi_i(c)\pi^*(C^*) \quad (19)$$

for all n, c and C^* . The customer arriving at epoch \hat{t}_{ni} enters W_i via node i independently with probability $\alpha_i / \left(\sum_{k \in W_i} \hat{\lambda}_k\right)$ for all n .

Thus, the points $\{t_{ni}, n = \pm 1, \pm 2, \dots\}$ are independently selected from the points $\{\hat{t}_{ni}, n = \pm 1, \pm 2, \dots\}$, and so from (19) we conclude that

$$P\{c_i(t_{ni}-) = c, C^*(t_{ni}-) = C^*\} = \psi_i(c)\pi^*(C^*) \quad (20)$$

for all n, i, c and C^* .

For the customer arriving at node i at epoch t_{ni} , let S_{ni} be the customer's total sojourn time at node i . Since service times at node i have an exponential distribution, the distribution of S_{ni} is completely determined by $c_i(t_{ni}) = c_i(t_{ni}-) + 1$. Thus, it follows that the sojourn times $\{S_{ni}, n = \pm 1, \pm 2, \dots\}$ have a common

exponential distribution with parameter $\nu_i = \mu_i - \alpha_i$ for $i = 1, 2, \dots, N$.
Moreover, by virtue of (20),

$$\begin{aligned} P\{S_{ni} \leq x, c_i(t_{ni}) = c+1, C^*(t_{ni}-) = C^*\} &= \\ \int_0^x P\{S_{ni} \in dy | c_i(t_{ni}) = c+1, C^*(t_{ni}-) = C^*\} \psi_i(c) \pi^*(C^*) &= \\ \int_0^x P\{S_{ni} \in dy | c_i(t_{ni}) = c+1\} \psi_i(c) \pi^*(C^*) &= \\ P\{S_{ni} \leq x, c_i(t_{ni}) = c+1\} \cdot \pi^*(C^*) &, \end{aligned}$$

where

$$P\{S_{ni} \leq x, c_i(t_{ni}) = c+1\} = \psi_i(c) \int_0^x \mu_i \frac{(\mu_i y)^c}{c!} e^{-\mu_i y} dy .$$

Hence, summing on c we have that

$$P\{S_{ni} \leq x, C^*(t_{ni}-) = C^*\} = P\{S_{ni} \leq x\} \cdot \pi^*(C^*) \quad (21)$$

for all n, i, C^* and x .

Next, let $E_i(s, t]$ be the total number of departures (external plus internal network transfers) from node i over the time interval $(s, t]$. Since $V_{(i)}$ is an exit set we know that the stream E_i is a Poisson process with rate α_i . We now observe that, conditioned on

S_{ni} , the total flow of customers through node i over the random time interval $(t_{ni}, t_{ni} + S_{ni})$ is also a Poisson process with rate α_i .¹ For this it suffices to show that for any $x > 0$ and any non-negative integer κ

$$P\{E_i(t_{ni}, t_{ni} + y) = \kappa | S_{ni} \in dx\} = \frac{e^{-\alpha_i y} (\alpha_i y)^\kappa}{\kappa!}, \quad 0 < y < x. \quad (22)$$

Now, the left side of (22) equals

$$\begin{aligned} \sum_{c > \kappa} P\{E_i(t_{ni}, t_{ni} + y) = \kappa, c_i(t_{ni}) = c | S_{ni} \in dx\} = \\ \sum_{c > \kappa} \frac{P\{E_i(t_{ni}, t_{ni} + y) = \kappa, S_{ni} \in dx | c_i(t_{ni}) = c\} \cdot P\{c_i(t_{ni}) = c\}}{P\{S_{ni} \in dx\}} = \\ \sum_{c > \kappa} \frac{\frac{(\alpha_i y)^\kappa e^{-\alpha_i y}}{\kappa!} \cdot \frac{\mu_i (\mu_i [x-y])^{c-\kappa-1} e^{-\mu_i (x-y)}}{(c - \kappa - 1)!} dx \cdot \psi_i(c-1)}{v_i e^{-v_i x} dx} = \\ \frac{(\alpha_i y)^\kappa}{\kappa!} e^{-\alpha_i y} \sum_{c > \kappa} \frac{(\alpha_i [x-y])^{c-\kappa-1}}{(c - \kappa - 1)!}, \end{aligned}$$

and this last expression equals the right side of (22).

¹This same phenomenon has been observed by R. L. Disney and co-workers.

5. THE PROCESS $\{C^*(t_{ni} + S_{ni} + t), t \geq 0\}$

In this section we observe that the process $\{C^*(t_{ni} + S_{ni} + t), t \geq 0\}$ evolves as a Markov process with the same (stationary) transition probability function, say p^* , as the process $\{C^*(s), -\infty < s < \infty\}$, and with $C^*(t_{ni} + S_{ni})$ independent of $c_i(t_{ni} + S_{ni})$ and of S_{ni} .

For the customer arriving at node i at epoch t_{ni} , let δ_{ni} be the node visited by this customer upon departing node i . Either $\delta_{ni} = 0$, in which case the customer exits the system from node i , or $\delta_{ni} = j$ for some node j in W_i^* . For j in W_i^* let $E_j^* = (e_k, k \text{ in } W_i^*)$ with $e_j = 1$ and $e_k = 0$ for $k \neq j$.

Now let $Z^*(s) = C^*(s-)$. We have

$$P\{c_i(t_{ni} + S_{ni}) = c, Z^*(t_{ni} + S_{ni}) = C^*, S_{ni} \leq x\} = \\ \sum_{D^*} \int_0^x P\{c_i(t_{ni} + y) = c, Z^*(t_{ni} + y) = C^* | Z^*(t_{ni}) = D^*, S_{ni} \in dy\} \cdot \\ P\{Z^*(t_{ni}) = D^*, S_{ni} \in dy\}$$

Now (22) implies that, conditioned on S_{ni} , node i contributes to the Poisson external input stream of the Jackson network W_i^* at rate $\alpha_i \sum_{W_i^*} p_{ij}$ over the random time interval $(t_{ni}, t_{ni} + S_{ni})$. Moreover by virtue of (21) the sojourn time S_{ni} and the vector $Z^*(t_{ni})$ are independent. And, the total stream of arrivals to node i is Poisson

with rate α_i . Hence, the integral above is equal to

$$\int_0^x \left[e^{-\alpha_i y} \frac{(\alpha_i y)^c}{c!} p^*(D^*, C^*, y) \right] \pi^*(D^*) v_i e^{-v_i y} dy.$$

Thus,

$$\begin{aligned} P\{c_i(t_{ni} + S_{ni}) = c, Z^*(t_{ni} + S_{ni}) = C^*, S_{ni} \leq x\} = \\ \int_0^x e^{-\alpha_i y} \frac{(\alpha_i y)^c}{c!} \left[\sum_{D^*} \pi^*(D^*) p^*(D^*, C^*, y) \right] v_i e^{-v_i y} dy = \end{aligned} \quad (23)$$

$$\psi_i(c) \pi^*(C^*) \int_0^x \mu_i \frac{(\mu_i y)^c}{c!} e^{-\mu_i y} dy,$$

where we have applied (7) to the transition function p^* and the equilibrium distribution π^* . Letting $x \rightarrow +\infty$ in (23) we see that $c_i(t_{ni} + S_{ni})$ and $Z^*(t_{ni} + S_{ni})$ are independent and have joint distribution $\psi_i(c) \pi^*(C^*)$. Summing on c in (22) we find that $Z^*(t_{ni} + S_{ni})$ and S_{ni} are independent.

We consider next the vector $C^*(t_{ni} + S_{ni})$. Now

$$\begin{aligned} P\{S_{ni} \leq x, C^*(t_{ni} + S_{ni}) = C^*\} &= P\{S_{ni} \leq x, Z^*(t_{ni} + S_{ni}) = C^*, \delta_{ni} = 0\} \\ &+ \sum_{W_j^*} P\{S_{ni} \leq x, Z^*(t_{ni} + S_{ni}) = C^* - E_j^*, \delta_{ni} = j\} = \\ &P\{S_{ni} \leq x\} \cdot \gamma_i(C^*) \end{aligned} \quad (24)$$

where

$$\gamma_i(C^*) = q_i \pi^*(C^*) + \sum_{W_i^*} p_{ij} \pi^*(C^* - E_j^*) \quad (25)$$

Now

$$\sum_{C^*} \pi^*(C^* - E_j^*) = \sum_{C^* = D^* + E_j^*} \pi^*(C^* - E_j^*) = \sum_{D^*} \pi^*(D^*) = 1$$

for any j in W_i^* . Moreover,

$$q_i + \sum_{W_i^*} p_{ij} = 1.$$

Hence, γ_i is a probability distribution on the states of $\{C^*(s), -\infty < s < +\infty\}$, and we conclude from (25) that both S_{ni} and $c_i(t_{ni} + S_{ni})$ are independent of $C^*(t_{ni} + S_{ni})$ and that γ_i is the distribution of $C^*(t_{ni} + S_{ni})$.

We now consider the process $\{C^*(t_{ni} + S_{ni} + t), t \geq 0\}$. Let \hat{p} denote the (stationary) transition probability function of the process $\{\hat{C}(s), -\infty < s < +\infty\}$. The epoch of departure $t_{ni} + S_{ni}$ from node i is a stopping time for process $\hat{C}(\cdot)$. And, the pure jump process $\hat{C}(\cdot)$ enjoys the strong Markov property; cf. Breiman [2, pp. 323, 328]. Then, for $u, x > 0$

$$P\{\hat{C}(t_{ni} + S_{ni}) = (d, D^*), \hat{C}(t_{ni} + S_{ni} + u) = (c, C^*)\} =$$

$$P\{C(t_{ni} + S_{ni}) = (d, D^*)\} \cdot \hat{p}((d, D^*), (c, C^*), u) =$$

(26)

$$\frac{\gamma_i(D^*)}{\pi^*(D^*)} \left[\psi_i(d) \pi^*(D^*) \hat{p}((d, D^*), (c, C^*), u) \right] =$$

$$\frac{\gamma_i(D^*)}{\pi^*(D^*)} P\{\hat{C}(0) = (d, D^*), \hat{C}(u) = (c, C^*)\},$$

where we have invoked (23) and (24), followed by the (ordinary) Markov property of the process $\hat{C}(\cdot)$. Summing on d and on c in (26), we find that

$$P\{C^*(t_{ni} + S_{ni}) = D^*, C^*(t_{ni} + S_{ni} + u) = C^*\} =$$

$$\frac{\gamma_i(D^*)}{\pi^*(D^*)} \cdot P\{C^*(0) = D^*, C^*(u) = C^*\} \quad (27)$$

$$\gamma_i(D^*) \cdot p^*(D^*, C^*, u),$$

using the (ordinary) Markov property of the process $\{C^*(s), -\infty < s < +\infty\}$.

Now suppose $u, v, x > 0$. By the conditional independence of the past and the future given the state of the process $\hat{C}(\cdot)$ at the stopping time $t_{ni} + S_{ni} + u$, cf. [3, p. 316], we have

$$P\{\hat{C}(t_{ni} + S_{ni}) = (d, D^*), \hat{C}(t_{ni} + S_{ni} + u) = (c, C^*), \hat{C}(t_{ni} + S_{ni} + u + v) = (b, B^*)\} =$$

$$P\{\hat{C}(t_{ni} + S_{ni}) = (d, D^*), \hat{C}(t_{ni} + S_{ni} + u) = (c, C^*)\} \cdot \hat{p}((c, C^*), (b, B^*), v) =$$

(28)

$$\frac{\gamma_i(D^*)}{\pi^*(D^*)} P\{\hat{C}(0) = (d, D^*), \hat{C}(u) = (c, C^*)\} \cdot \hat{p}((c, C^*), (b, B^*), u) =$$

$$\frac{\gamma_i(D^*)}{\pi^*(D^*)} P\{\hat{C}(0) = (d, D^*), \hat{C}(u) = (c, C^*), \hat{C}(u + v) = (b, B^*)\} ,$$

also making use of (26) and the Markov property of the process $\hat{C}(\cdot)$.

Summing on d and on c and on b in (28), we find that

$$P\{C^*(t_{ni} + S_{ni}) = D^*, C(t_{ni} + S_{ni} + u) = C^*, C^*(t_{ni} + S_{ni} + u + v) = B^*\} =$$

$$\frac{\gamma_i(D^*)}{\pi^*(D^*)} P\{C^*(0) = D^*, C^*(u) = C^*, C^*(u + v) = B^*\} = \quad (29)$$

$$\gamma_i(D^*) \cdot p^*(D^*, C^*, u) p^*(C^*, B^*, v) .$$

Proceeding by induction, we find that, for all $0 = u_0 < u_1 < u_2 < \dots < u_m < +\infty$ and $x > 0$ and all states $C_0^*, C_1^*, C_2^*, \dots, C_m^*$ of the process $\{C^*(s), -\infty < s < +\infty\}$, we have

$$P \left\{ \bigcap_{r=0}^m \{C^*(t_{ni} + S_{ni} + u_r) = C_r^*\} \right\} = \quad (30)$$

$$\gamma_i(C_0^*) \cdot \prod_{r=1}^m p^*(C_{r-1}^*, C_r^*, u_r - u_{r-1}) .$$

And, as a consequence of (30), we conclude that $\{C^*(t_{ni} + S_{ni} + t), t \geq 0\}$ evolves as a Markov process with initial distribution γ_i^* and stationary transition probability function p^* .

6. SOJOURN TIMES $\{T_{ni}, n = \pm 1, \pm 2, \dots\}$

In a manner similar to that of (9) of Section 1, let

$$t_{ni} + T_{ni}$$

be the epoch of departure from the network for the customer arriving at node i at epoch t_{ni} (external arrival to node i or internal transfer to node i). Since the set of nodes W_i (as defined in Section 4) is a Jackson network in equilibrium, the sojourn times $\{T_{ni}, n = \pm 1, \pm 2, \dots\}$ are identically distributed; let g_i denote the Laplace transform of their common distribution. In this section we use the strong Markov property, in conjunction with the developments in Section 5, to derive an expression for the transform g_i which leads to the results (10a) and (10b).

Let $Y_{ni} = T_{ni} - S_{ni}$ and $\theta_1, \theta_2 \geq 0$. By the conditional independence of the past and the future of the process $\{\hat{C}(t), -\infty < t < +\infty\}$ at the stopping time $t_{ni} + S_{ni}$, we have that

$$\begin{aligned} E \left\{ e^{-\theta_1 S_{ni}} e^{-\theta_2 Y_{ni}} \mid c_i(t_{ni} + S_{ni}) = c \right\} = \\ E \left\{ E \left[e^{-\theta_1 S_{ni}} \mid \hat{C}(t_{ni} + S_{ni}) \right] \cdot E \left[e^{-\theta_2 Y_{ni}} \mid \hat{C}(t_{ni} + S_{ni}) \right] \cdot 1 \mid c_i(t_{ni} + S_{ni}) = c \right\} \end{aligned} \quad (31)$$

Since $c_i(t_{ni} + S_{ni})$ and $\hat{C}^*(t_{ni} + S_{ni})$ are independent

$$E \left[e^{-\theta_1 S_{ni}} | \hat{C}(t_{ni} + S_{ni}) \right] = E \left[e^{-\theta_1 S_{ni}} | c_i(t_{ni} + S_{ni}) \right]$$

with probability one; and a version of the conditional expectation on the right side immediately above is clearly

$$\left(\frac{\mu_i}{\mu_i + \theta_1} \right)^{c_i(t_{ni} + S_{ni}) + 1}$$

Hence, the right side of (31) is equal to

$$\left(\frac{\mu_i}{\mu_i + \theta_1} \right)^{c+1} E \left\{ E \left[e^{-\theta_2 Y_{ni}} \right]_{c_i(t_{ni} + S_{ni}) = c} \mid c_i(t_{ni} + S_{ni}) \right\}$$

The expectation immediately above is

$$E \left\{ E \left[e^{-\theta_2 Y_{ni}} \mid c_i(t_{ni} + S_{ni}) \right] \cdot 1_{\{c_i(t_{ni} + S_{ni}) = c\}} \right\}$$

which in turn equals

$$q_i \psi_i(c) + \sum_{W_i^*} E \left\{ E \left[e^{-\theta_2 Y_{ni}} \mid c_i(t_{ni} + S_{ni}) \right] \cdot 1_{\{c_i(t_{ni} + S_{ni}) = c, S_{ni} = j\}} \right\}$$

If the customer is routed to node j in W_i^* then the customer enters the Jackson network W_i^* as an external arrival in the sense that

$Z^*(t_{ni} + S_{ni})$ has distribution π^* and $Z^*(t_{ni} + S_{ni})$, δ_{ni} and $c_i(t_{ni} + S_{ni})$ are all independent of one another. The process $\{C^*(t_{ni} + S_{ni} + t), t \geq 0\}$ evolves as a Markov process with stationary transition probability function p^* , and upon entering the set of nodes W_i^* a customer remains within W_i^* until exiting the system. Moreover, the distribution of $c_i(t_{nj})$ is ψ_i for all n and j in W_i^* , by virtue of Section 3 applied to the process $\hat{C}(\cdot)$. We conclude that

$$E \left\{ E \left[e^{-\theta_2 Y_{ni}} | c_i(t_{ni} + S_{ni}) \right] \cdot 1_{\{c_i(t_{ni} + S_{ni}) = c, \delta_{ni} = j\}} \right\} =$$

$$p_{ij} \psi_i(c) \cdot E \left[e^{-\theta_2 T_{nj}} | c_i(t_{nj}) = c \right] =$$

$$p_{ij} E \left\{ e^{-\theta_2 T_{nj}} 1_{\{c_i(t_{nj}) = c\}} \right\}$$

for all j in W_i^* . Thus,

$$E \left\{ e^{-\theta_1 S_{ni}} e^{-\theta_2 Y_{ni}} 1_{\{c_i(t_{ni} + S_{ni}) = c\}} \right\} =$$

$$\left(\frac{\mu_i}{\mu_i + \theta_1} \right)^{c+1} \left[q_i \psi_i(c) + \sum_{W_i^*} p_{ij} E \left\{ e^{-\theta_2 T_{nj}} 1_{\{c_i(t_{nj}) = c\}} \right\} \right]$$

Summing on c we find that

$$E \left\{ e^{-\theta_1 S_{ni}} e^{-\theta_2 Y_{ni}} \right\} = \quad (32)$$

$$q_i \frac{v_i}{v_i + \theta_1} + \sum_{j=1}^N p_{ij} \sum_{c=0}^{\infty} \left(\frac{\mu_i}{\mu_i + \theta_1} \right)^{c+1} E \left\{ e^{-\theta_2 T_{nj}} 1_{\{c_i(t_{nj}) = c\}} \right\}$$

for all $n = \pm 1, \pm 2, \dots$ and all $i = 1, 2, \dots, N$. Putting $\theta_1 = 0$ and $\theta_2 = \theta$ in (32) we find that

$$E \left\{ e^{-\theta Y_{ni}} \right\} = q_i + \sum_{j=1}^N p_{ij} g_j(\theta) \quad (33)$$

If the acyclic network also has a "tree-like" structure, then T_{nj} and $c_i(t_{nj})$ are independent for j in W_i^* , whence

$$E \left\{ e^{-\theta T_{nj}} 1_{\{c_i(t_{nj}) = c\}} \right\} = \psi_j(c) g_j(\theta) \quad .$$

Thus, if the network has a "tree-like" structure, then on putting $\theta = \theta_1 = \theta_2$ in (32) we obtain

$$g_i(\theta) = \frac{v_i}{v_i + \theta} \cdot \left[q_i + \sum_{j=1}^N p_{ij} g_j(\theta) \right] \quad (34)$$

for $i = 1, 2, \dots, N$.

7. THE SOJOURN TIMES $\{T_n, n = \pm 1, \pm 2, \dots\}$

For the external network arrival at epoch t_n let δ_n be the node through which the customer enters the network. The sojourn times $\{T_n, n = \pm 1, \pm 2, \dots\}$ are identically distributed, and if h is the Laplace transform of their common distribution then

$$h(\theta) = \sum_{i=1}^N E \left\{ e^{-\theta T_n} 1_{\{\delta_n=i\}} \right\} .$$

If $\bar{\lambda}_i = \lambda_i / (\lambda_1 + \lambda_2 + \dots + \lambda_N)$ then $P\{\delta_n = i\} = \bar{\lambda}_i$. By virtue of Section 6 we must have

$$E \left\{ e^{-\theta T_n} 1_{\{\delta_n=i\}} \right\} = \bar{\lambda}_i g_i(\theta) ,$$

and so

$$h(\theta) = \sum_{i=1}^N \bar{\lambda}_i g_i(\theta) . \quad (35)$$

When each transform g_i satisfies (34), it is a straightforward matter to solve for $h(\theta)$. Begin by rewriting (34) as

$$[(v_i + \theta)/v_i] g_i(\theta) = q_i + \sum_{j=1}^N p_{ij} g_j(\theta)$$

for $i = 1, 2, \dots, N$. If $g(\theta)$ is a column vector whose i th entry

is $g_i(\theta)$ and if $\mathcal{Q}(\theta)$ and q are as defined in Section 1, then (34) is equivalent to

$$(\mathcal{Q}(\theta) - \mathcal{P})g(\theta) = q. \quad (36)$$

The matrix $\mathcal{Q}(\theta)$ is clearly invertible for any $\theta \geq 0$ and so we can write

$$\mathcal{Q}(\theta) - \mathcal{P} = (I - \mathcal{P}[\mathcal{Q}(\theta)]^{-1})\mathcal{Q}(\theta). \quad (37)$$

Each diagonal element of the diagonal matrix $[\mathcal{Q}(\theta)]^{-1}$ is in $(0,1]$, and so it follows from (37) that $(\mathcal{Q}(\theta) - \mathcal{P})$ is invertible for any $\theta \geq 0$. With $\bar{\lambda}$ a row vector whose i th entry is $\bar{\lambda}_i$, we now see that the right side of (35) is equal to

$$\bar{\lambda} [\mathcal{Q}(\theta) - \mathcal{P}]^{-1} q.$$

The proof of (10b) is now complete.

Applying Cramer's rule to (36), we see that when (34) holds each transform g_i is a rational function whose denominator is a polynomial in θ of degree at most N . Thus, h is the transform of a mixture of exponential distributions where (34) holds.

8. THE MOMENTS OF TOTAL SOJOURN TIME

Let T_1, T_2, \dots, T_N, T be random variables whose distributions have Laplace transforms g_1, g_2, \dots, g_N, h , respectively. Any customer never visits a node more than once, and under equilibrium conditions each node is a $M/M/1$ queue in equilibrium. Thus, each of the variables T_1, T_2, \dots, T_N, T has finite moments of all orders. It follows from (33) that the residual sojourn time Y_{ni} has mean $\sum_{j=1}^N p_{ij} E\{T_j\}$ for all n and i , and so

$$E\{T_i\} = 1/v_i + \sum_{j=1}^N p_{ij} E\{T_j\} \quad (38)$$

for $i = 1, 2, \dots, N$. Then

$$\begin{aligned} \sum_{i=1}^N \alpha_i E\{T_i\} &= \sum_{i=1}^N \alpha_i / v_i + \sum_{i=1}^N \alpha_i \sum_{j=1}^N p_{ij} E\{T_j\} \\ &= \sum_{i=1}^N \alpha_i / v_i + \sum_{j=1}^N E\{T_j\} \sum_{i=1}^N \alpha_i p_{ij} \\ &= \sum_{i=1}^N \alpha_i / v_i + \sum_{j=1}^N (\alpha_j - \lambda_j) E\{T_j\}, \end{aligned}$$

with the last equality holding by virtue of (3a). Hence,

$$\sum_{i=1}^N \lambda_i E\{T_i\} = \sum_{i=1}^N \alpha_i / v_i . \quad (39)$$

Noting that $v_i = \mu_i(1 - \rho_i)$, it now follows from (35) and (39) that

$$E\{T\} = \left(\sum_{i=1}^N \lambda_i \right)^{-1} \sum_{i=1}^N \frac{\rho_i}{1 - \rho_i} .$$

Moreover, if v is a column vector whose i th entry is $1/v_i$, then it follows from (38) that $E\{T_i\}$ is the i th entry of the column vector $[I - P]^{-1}v$ for $i = 1, 2, \dots, N$.

When each transform g_i satisfies (34), it is possible to compute the higher moments of total sojourn time by a straightforward recursive procedure. For example, taking the second derivative with respect to θ of both sides of (34) and then setting $\theta = 0$ gives

$$E\{T_i^2\} = \sum_{j=1}^N p_{ij} E\{T_j^2\} + 2 \left(\sum_{j=1}^N p_{ij} E\{T_j\} \right) / v_i + 2/v_i^2$$

for $i = 1, 2, \dots, N$, which, in view of (38), is equivalent to

$$E\{T_i^2\} = 2E\{T_i\}/v_i + \sum_{j=1}^N p_{ij} E\{T_j^2\} \quad (41)$$

for $i = 1, 2, \dots, N$. The same approach used in deriving (39) yields

$$\sum_{i=1}^N \lambda_i E\{T_i^2\} = 2 \sum_{i=1}^N \alpha_i E\{T_i\} / v_i \quad (42)$$

Thus, we have

$$E\{T^2\} = 2 \left(\sum_{i=1}^N \lambda_i \right)^{-1} \sum_{i=1}^N \frac{\rho_i}{1 - \rho_i} E\{T_i\} \quad (43)$$

when (34) holds. Also, it follows from (41) that if σ is a column vector whose i th entry is $2E\{T_i\}/v_i$, then $E\{T_i^2\}$ is the i th entry of the column vector $[I - \rho]^{-1} \sigma$ for $i = 1, 2, \dots, N$. This method of computing $E\{T_i\}$, $E\{T_i^2\}$, for $i = 1, 2, \dots, N$, and $E\{T\}$, $E\{T^2\}$ can clearly be continued to obtain further higher moments when (34) holds.

Methods for computing mean sojourn times in some other queueing network models are given in [12] and [13].

REFERENCES

- [1] BEUTLER, F. J. and MELAMED, B. (1977). Decomposition and Customer Streams of Feedback Queueing Networks in Equilibrium. Technical Report 77-1, Computer, Information and Control Engineering Program. The University of Michigan.
- [2] BREIMAN, L. (1968). Probability. Addison-Wesley, Reading, Mass.
- [3] CHUNG, K. L. (1974). A Course on Probability Theory. 2nd Ed. Holden-Day, San Francisco.
- [4] IGLEHART, D. L. and SHEDLER, G. S. (1978). Regenerative Simulation of Response Times in Networks of Queues. J. Assoc. Comput. Mach. 25, pp. 449-460.
- [5] IGLEHART, D. L. and SHEDLER, G. S. (1978). Simulation of Response Times in Finite Capacity Open Networks of Queues, Operations Res. 26, pp. 896-914.
- [6] IGLEHART, D. L. and SHEDLER, G. S. (1978). Regenerative Simulation of Response Times in Networks of Queues, II: Multiple Job Types. IBM Research Report RJ 2256.
- [7] JACKSON, J. R. (1957). Networks of Waiting Lines. Operations Res. 5, pp. 518-521.
- [8] LAVENBERG, S. S. (1978). Regenerative Simulation of Queueing Networks. IBM Research Report RC 7087.
- [9] LEMOINE, A. J. (1977). Networks of Queues - A Survey of Equilibrium Analysis. Management Sci. 24, pp. 464-481.
- [10] LEMOINE, A. J. (1978). Networks of Queues - A Survey of Weak Convergence Results. Management Sci. 24, pp. 1175-1193.
- [11] REICH, E. (1963). Note on Queues in Tandem. Ann. Math Statist. 34, pp. 338-341.
- [12] REISER, M. and LAVENBERG, S. S. (1978). Mean Value Analysis of Closed Multichain Queueing Networks. IBM Research Report RC 7023.
- [13] SCHASSBERGER, R. (1978). Mean Sojourn Times in Insensitive Generalized Semi-Markov Schemes. Technical Report, Department of Mathematics and Statistics, University of Calgary.
- [14] WALRAND, J. and VARAIYA, P. (1978). The Outputs of Jacksonian Networks Are Poissonian. Memorandum No. ERL-M78/60, Electronics Research Laboratory, University of California, Berkeley.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|--|-----------------------|--|
| 1. REPORT NUMBER 14 712-1 | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER 9 |
| 4. TITLE (and Subtitle) 6 On Total Sojourn Time in Acyclic Jackson Networks. | | 5. TYPE OF REPORT & PERIOD COVERED Technical Report |
| 7. AUTHOR(s) 10 Austin J. Lemoine | | 6. PERFORMING ORG. REPORT NUMBER |
| 8. PERFORMING ORGANIZATION NAME AND ADDRESS CONTROL ANALYSIS CORPORATION 800 Welch Road Palo Alto, CA 94304 | | 9. CONTRACT OR GRANT NUMBER(s) 15 NR0014-78-C-0712 |
| 10. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Operations Research Program, Code 434 Arlington, Virginia 22217 | | 11. PROGRAM ELEMENT PROJECT, TASK AREA & WORK UNIT NUMBERS NR-047-106 |
| 12. REPORT DATE 11 Jan 1979 | | 13. NUMBER OF PAGES 38 pages 12 38 p. |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | | 15. SECURITY CLASS. (of this report) Unclassified |
| 16. DISTRIBUTION STATEMENT (of this Report) Distribution of this document is unlimited. | | |
| 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) | | |
| 18. SUPPLEMENTARY NOTES | | |
| 19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Single-Server Queues, Acyclic Jackson Networks, Sojourn Time Distribution, Infinite Capacity Open Network Models | | |
| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number) (Abstract appears on page i) | | |

407383

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 65 IS OBSOLETE

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)